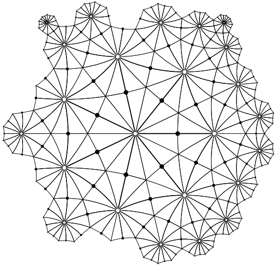
2-solvable Belyĭ maps



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Outline

Belyĭ's Theorem

Theorem (G.V. Belyĭ 1979)

A smooth projective curve X over $\mathbb{C}$ *can be defined over* $\overline{\mathbb{Q}}$ *if and only if there exists a branched covering of compact connected Riemann surfaces* $\varphi : X \to \mathbb{P}^1$ *unramified (unbranched) above* $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

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In the 1980s, Grothendieck described a bijection between Belyĭ maps and *dessins d'enfants*. $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on these sets.

A Zoo of Bijections

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$\left\{ \text{Transitive permutation Triples } (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3 \text{ with } \sigma_\infty \sigma_1 \sigma_0 = 1 \right\}$

∼

$\left\{ \text{Index } d \text{ triangle subgroups } \Gamma \leq \Delta(a, b, c) \right\}$

∼

$\left\{ \text{dessins with } d \text{ edges} \right\}$

∼

$\left\{ \text{Degree } d \text{ function field extensions } K \supseteq \mathbb{C}(z) \text{ with discriminant supported above } 0, 1, \infty \right\}$

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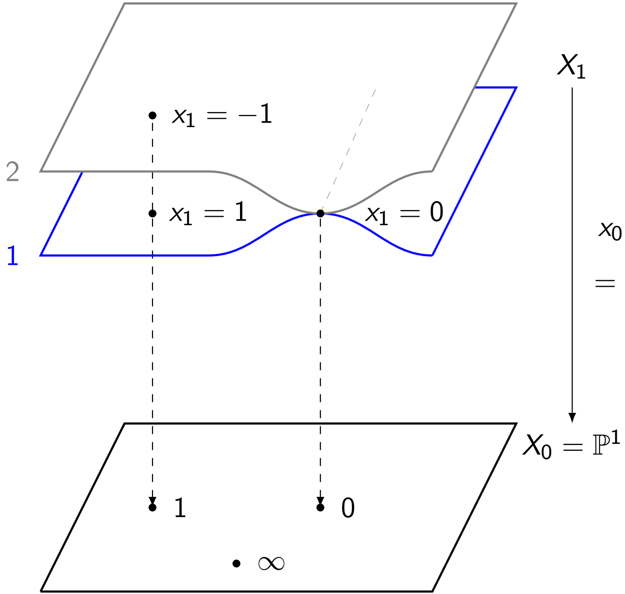
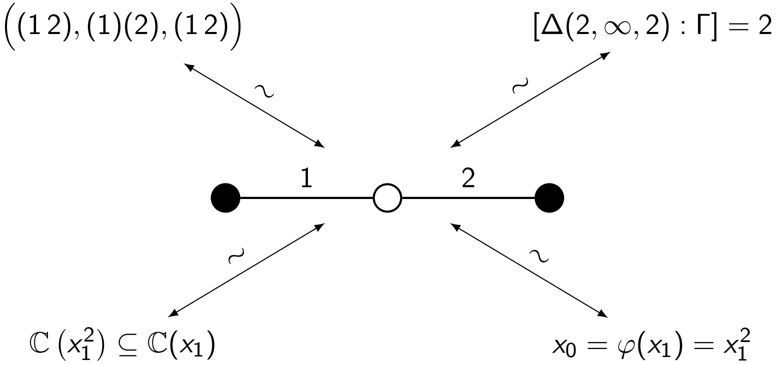
All up to the appropriate version of equivalence in each category.



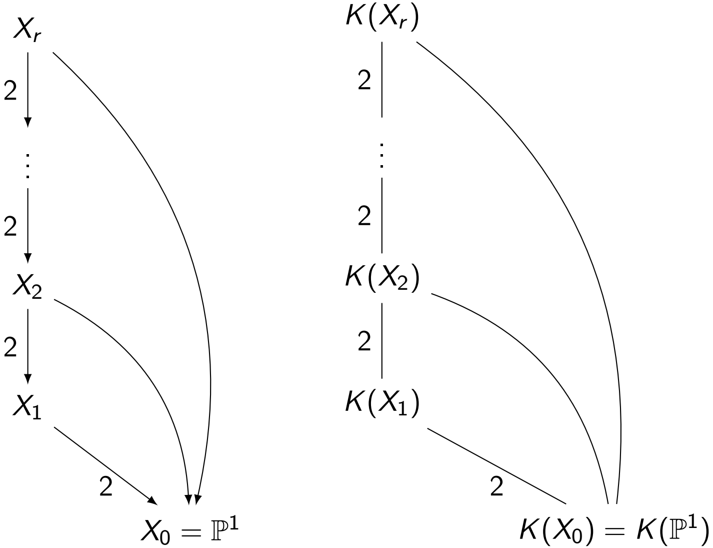
$$x_0 = \varphi(x_1) = x_1^2$$
$$= (x_1 + 1)(x_1 - 1) + 1$$

Example ☕



$\big((1\,2),(1)(2),(1\,2)\big)$
$[\Delta(2,\infty,2):\Gamma]=2$
1
2
$\sim$
$\sim$
$\sim$
$\sim$
$\mathbb{C}\left(x_1^2\right)\subseteq\mathbb{C}(x_1)$
$x_0=\varphi(x_1)=x_1^2$

2-solvable (Galois) Belyĭ maps


X_r
2
$\vdots$
2
X_2
2
X_1
2
$X_0 = \mathbb{P}^1$
$K(X_r)$
2
$\vdots$
2
$K(X_2)$
2
$K(X_1)$
2
$K(X_0) = K(\mathbb{P}^1)$

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Theorem (Beckmann-Kazez 1989)

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Upshot:

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Upshot: Every 2-solvable Belyĭ curve we write down has good reduction away from $p = 2$.

Lifting example ☕

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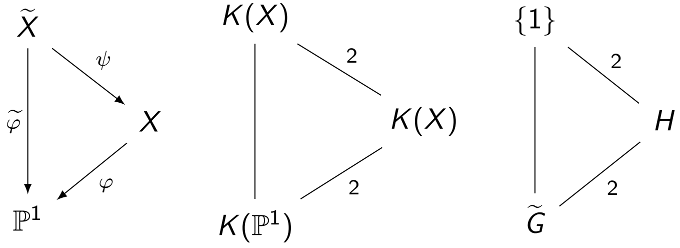

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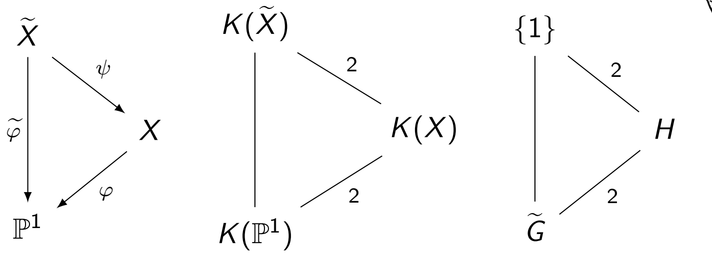

$G = \mathrm{Gal}(K(X)/K(\mathbb{P}^1))$ $\quad G \cong \left\langle \Big((1\,2), (1)(2), (1\,2) \Big) \right\rangle \leq S_2$

$\widetilde{G} = \mathrm{Gal}(K(\widetilde{X})/K(\mathbb{P}^1))$ $\quad \widetilde{G} \cong \langle \widetilde{\sigma} \rangle \leq S_4$

$H = \mathrm{Gal}(K(\widetilde{X})/K(X))$ $\quad H \cong \langle (1\,3)(2\,4) \rangle \leq S_4$

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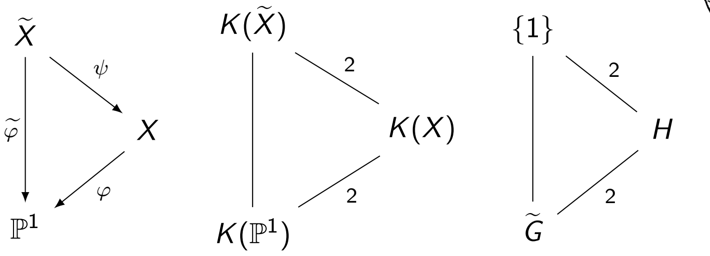

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$$\widetilde{\sigma} \xrightarrow{?} \sigma$$

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$$\sigma = (\sigma_0, \sigma_1, \sigma_\infty) = \big((1\,2), (1)(2), (1\,2)\big) \in S_2^3$$

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$f^{-1}(\sigma_0)$	$f^{-1}(\sigma_1)$	$f^{-1}(\sigma_\infty)$
$(1\,2)(3\,4)$	$(1)(2)(3)(4)$	$(1\,2)(3\,4)$
$(1\,4)(2\,3)$	$(1\,3)(2\,4)$	$(1\,4)(2\,3)$
$(1\,4\,3\,2)$		$(1\,4\,3\,2)$
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$\widetilde{G} \cong \mathbb{Z}/4\mathbb{Z}$
$\big((1\,4\,3\,2), (1)(2)(3)(4), (1\,2\,3\,4)\big)$
$\big((1\,4\,3\,2), (1\,3)(2\,4), (1\,4\,3\,2)\big)$

$\widetilde{G} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$\big((1\,2)(3\,4), (1\,4)(2\,3), (1\,3)(2\,4)\big)$

4T1-[4,2,4]-4-22-4-g1

$$(\sigma_0, \sigma_1, \sigma_\infty) = \Big((1\,4\,3\,2), (1\,3)(2\,4), (1\,4\,3\,2)\Big)$$

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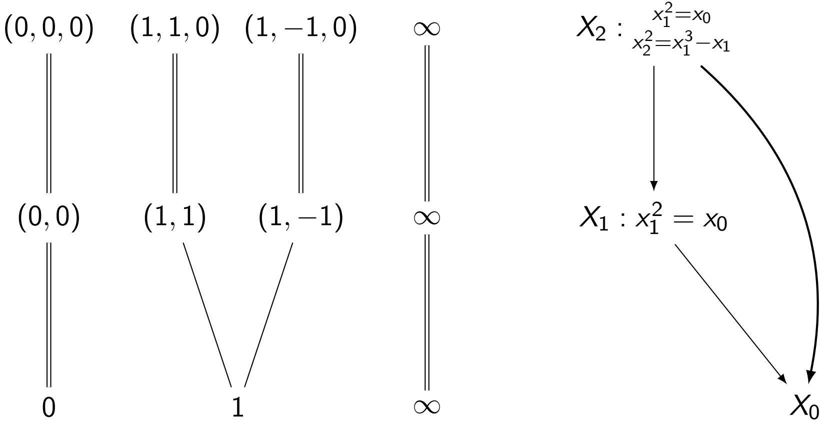

16T6-[8,8,4]-88-88-4444-g5

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We now exhibit a genus 5 Belyĭ map $\varphi : X \to \mathbb{P}^1$ defined by $x_0 \in K(X)$ with monodromy group $C_8 : C_2$.

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X is cut out by the following equations in $\mathbb{A}^5$:

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X is cut out by the following equations in $\mathbb{A}^5$:

$$
\begin{aligned}
x_1^2 &= x_0 \\
x_3x_4^2 &= x_1x_2 + x_1 + x_3^2 \\
x_3x_4^2 &= x_1 + x_1x_3^2 \\
x_3x_4^4 &= x_3^3 + 2x_1x_4^2 \\
2x_3^2x_4^4 &= x_2^2 + 2x_3^3x_4^2 + 2x_3^2 - 2x_3x_4^2 + 2x_4^4 + 1 \\
x_3^3 &= x_2x_3 + x_3^2x_4^2 - x_4^2 \\
x_3^2x_4^2 &= x_2x_4^2 + x_3^3 + x_3 \\
x_3^2x_4^4 &= x_3^4 + x_3^2 + x_4^4
\end{aligned}
$$

Acknowledgements

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- Sam Schiavone
- Jeroen Sijsling
- John Voight

Thanks for listening!

https://math.dartmouth.edu/~mjmusty/32.html