Dessins d'Enfants

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Abstract

We start with a general introduction to the theory of dessins d'enfants (children's drawings) in relation to number theory, geometry, and art. We then give an explicit example and describe current work in creating a database of these mathematical objects.

Motivation

What is a dessin d'enfant? A *dessin d'enfant* (or simply dessin) is a connected 2-colored graph with the property that each vertex has a cyclic ordering of the edges meeting it.

Why are they interesting? In [6] Alexander Grothendieck describes an action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins d'enfants. About this discovery, Grothendieck said: *I do not believe that a mathematical fact has ever struck me so strongly as this one.* For an explicit example of the Galois action see Figure 2. The impossibly complicated group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ embodies the entirety of Galois theory over the rational numbers. And yet somehow, Grothendieck's work provides a way to understand this complicated object in terms of mere children's drawings. More formally,

- The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of dessins d'enfants (cf. [6]).
- A curve X over $\mathbb{C}$ can be defined over $\overline{\mathbb{Q}}$ if and only if X admits a Belyĭ map (cf. [1, 2]).

While number theorists are led to study dessins for their own sake, the applications of modern number theory to cryptography and other areas makes the theoretical underpinnings of the subject important outside of the academic realm. Lastly, dessins have connections to hyperbolic geometry that can be visualized with intriguing pictures and art (see Figure 3).

The Upshot

Dessins d'enfants are mathematical objects that are of interest in algebraic number theory. Part of the fascination behind these objects is the prospect of understanding complex arithmetic objects with simple (and sometimes beautiful) pictures.

A Zoo of Bijections

Dessins have many equivalent formulations. In Figure 1 we summarize them in a diagram.

$$\left\{\begin{array}{l}\text{Transitive permutation}\\ \text{triples } (\sigma_0,\sigma_1,\sigma_\infty) \in S_d^3\\ \text{with } \sigma_\infty\sigma_1\sigma_0 = 1 \text{ up to}\\ \text{simultaneous conjugation}\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{l}\text{Isomorphism classes of}\\ \text{dessins with } d \text{ edges}\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{l}\text{Conjugacy classes of in-}\\ \text{dex } d \text{ triangle sub-}\\ \text{groups } \Gamma \leq \Delta(a,b,c)\end{array}\right\}$$

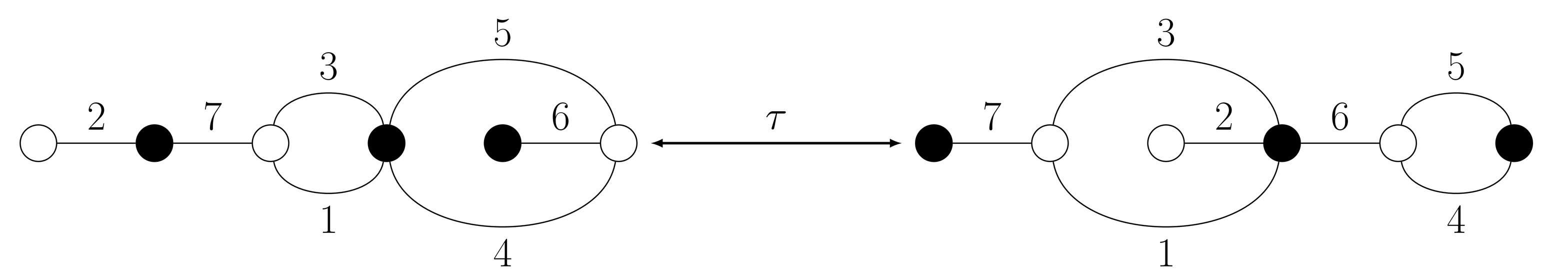
$$\left\{\begin{array}{l}\text{Isomorphism classes of degree } d \text{ func-}\\ \text{tion field extensions } K \supseteq \mathbb{C}(z) \text{ with}\\ \text{discriminant supported above } 0,1,\infty\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{l}\text{Isomorphism classes of}\\ \text{dessins with } d \text{ edges}\end{array}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{array}{l}\text{Isomorphism classes of}\\ \text{Belyĭ maps of degree } d\end{array}\right\}$$

Figure 1: A partial diagram of sets in bijection with the set of dessins of degree d.

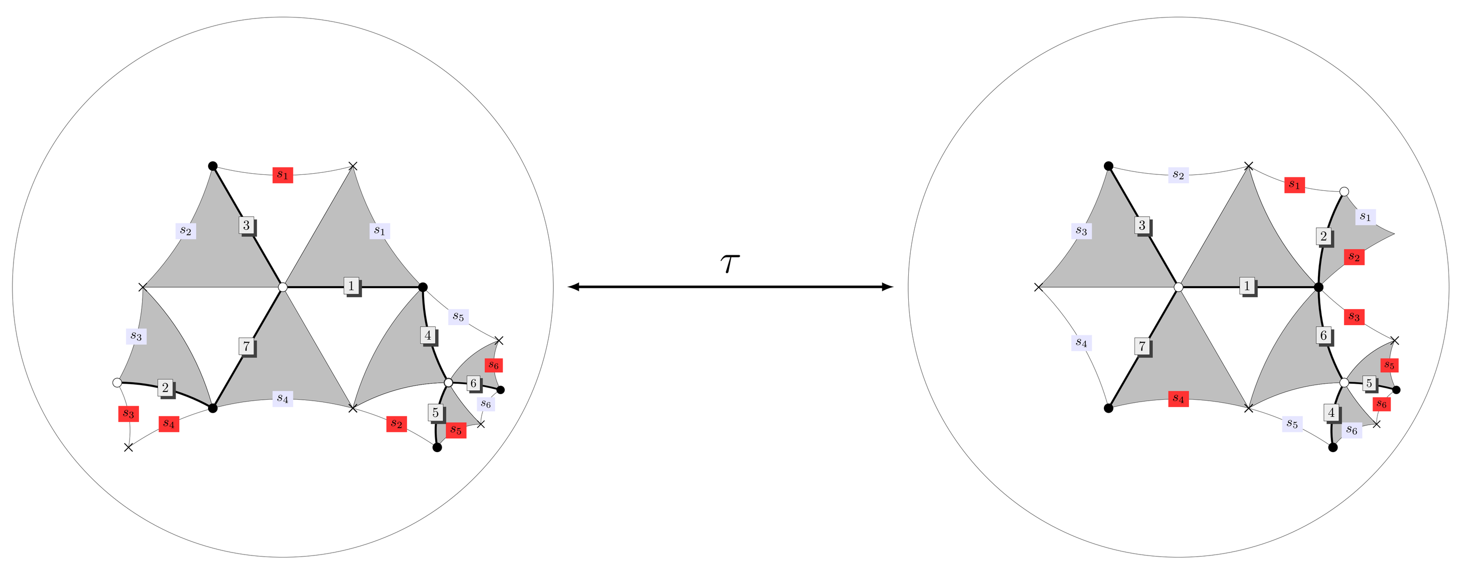
The Upshot

The collection of dessins is in correspondence with many different classes of objects. These correspondences allow us to attack problems using tools from a variety of different areas (e.g. group theory, field theory, algebraic geometry, and combinatorics) and to tailor our approach to each particular setting. In order to make a database, we use a description suited to calculations by computer (permutation triples). To view the situation from an artistic perspective, we can instead use a geometric description (triangle subgroups). See Figure 2 for an explicit example of how we can move between these categories.

Example 7T5-[3,4,4]-331-421-421-g0



$$\begin{array}{l}\sigma_0 = (1\,3\,7)(2)(4\,5\,6)\\ \sigma_1 = (1\,4\,5\,3)(2\,7)(6)\\ \sigma_\infty = (1\,2\,7\,5)(3)(4\,6)\end{array} \xleftrightarrow{\quad\tau\quad} \begin{array}{l}\sigma_0 = (1\,3\,7)(2)(4\,5\,6)\\ \sigma_1 = (1\,6\,3\,2)(4\,5)(7)\\ \sigma_\infty = (1\,7\,6\,4)(2\,3)(5)\end{array}$$



$$\phi = \underbrace{\left(\frac{1}{3087}\left(173\sqrt{7}+343\right)\right)}_{\lambda\in\mathbb{R}} \cdot \frac{x^3\left(x-\frac{1}{729}(68\sqrt{7}+236)\right)^1\left(x-\frac{1}{9}(20-4\sqrt{7})\right)^3}{\left(x-\frac{4}{21}(\sqrt{7}+3)\right)^2\left(x-\frac{4}{21}(\sqrt{7}+1)\right)^4}$$

$$\phi - 1 = \lambda \cdot \frac{\left(x-\frac{1}{189}(44\sqrt{7}+140)\right)^4\left(x-\frac{1}{7}(12\sqrt{7}-28)\right)^2\left(x-\frac{1}{14}(3\sqrt{7}+7)\right)^1}{\left(x-\frac{4}{21}(\sqrt{7}+3)\right)^2\left(x-\frac{4}{21}(\sqrt{7}+1)\right)^4}$$

Figure 2: Above we give an example of 2 dessins conjugate under the Galois action. From the top down we have the 2 dessins, their representative permutation triples, and corresponding fundamental domains as in Figure 1. At the bottom we have written down the Belyĭ map ϕ which is the same for both dessins up to the 2 choices for embedding $\mathbb{Q}\left(\sqrt{7}\right)$ into $\mathbb{R}$. The action of the Galois element τ, interchanging $\pm\sqrt{7}$, on the coefficients of ϕ induces the same interchanging action on the dessins, permutation triples, and fundamental domains above.

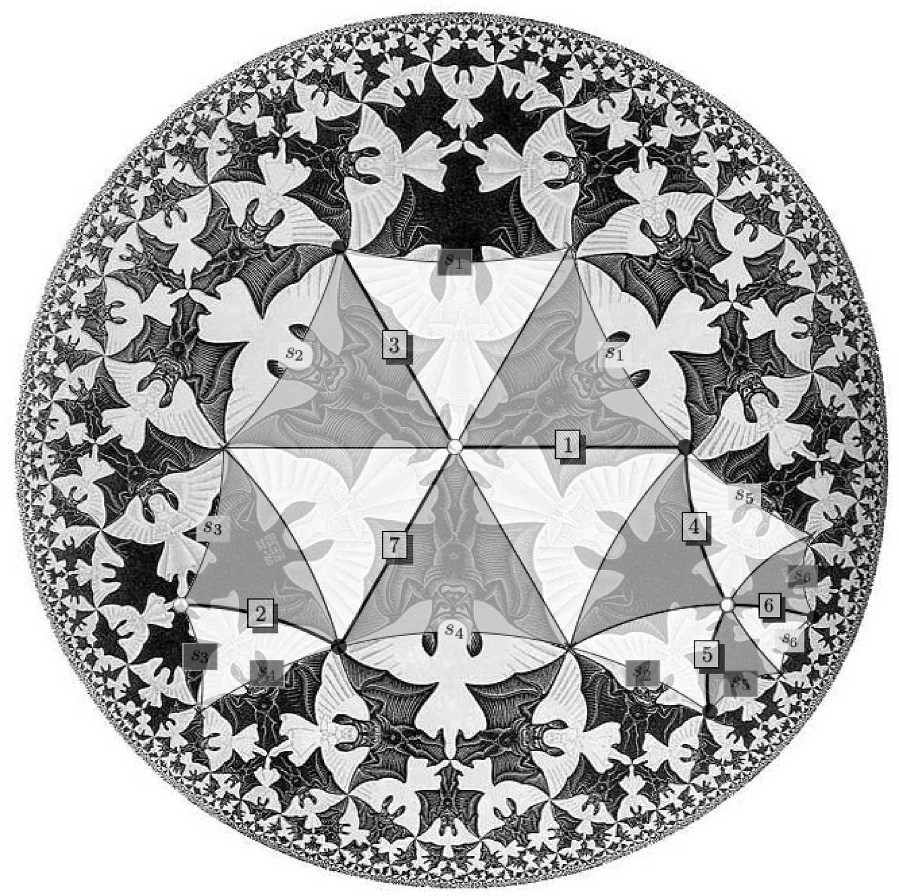


Figure 3: Escher's *Circle Limit IV* [5] with an overlay from Figure 2. The triangle subgroup (whose fundamental domain is pictured here) is an index 7 subgroup of the containing triangle group $\Delta(3,4,4)$ which is also the group of symmetries of Escher's underlying woodcut. For more on Escher's Circle Limit woodcuts and his collaboration with the mathematician HSM Coxeter see [4].

A Database of Belyĭ Maps

Ever since Grothendieck's work in the 1980s, number theorists have been trying to understand the mysterious action of the absolute Galois group on dessins. As far as the action of Galois is concerned, the objects we would like to compute are the Belyĭ maps (maps of algebraic curves) associated to dessins. Although Figure 1 tells us Belyĭ maps are in bijection with dessins, moving from the combinatorial description to equations can be difficult (cf. [10]). Even in [6], Grothendieck expresses this difficulty saying: *I doubt there is a uniform method for solving the problem by computer.* Nevertheless, a general purpose numerical method for computing Belyĭ maps was developed (cf. [7, 8, 11]). Using the computer algebra system `Magma` [3], our main result is the tabulation of a database consisting of all Belyĭ maps up to degree 9 using these numerical techniques. An example of the type of information we can get from this database is given in Table 1. More detailed information can be found in [9], and our database will be made available on the *L-functions and Modular Forms Database* at `www.lmfdb.org`.

degree	genus 0	genus 1	genus 2	genus 3	genus > 3	total
2	1	0	0	0	0	1
3	2	1	0	0	0	3
4	6	2	0	0	0	8
5	12	6	2	0	0	20
6	38	29	7	0	0	74
7	89	50	13	3	0	155
8	261	217	84	11	0	573
9	583	427	163	28	6	1207

Table 1: For degree up to 9, the total number of Belyĭ maps (dessins) of a given genus.

The Upshot

Number theorists are motivated to study dessins because they correspond to interesting algebraic equations. Finding these equations is a difficult task. Although a great deal of effort has gone into computing these equations for specific examples, relatively little work has been done towards computing *all* examples in a given range. Our main result here is a database of all examples up to a bound bringing us one step closer to understanding the Galois action on dessins.

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